Representations of quantum orders

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[1]

Abstract

We study finite dimensional algebras that appear as fibers of quantum orders over a given point of variety of center. We present the formula for the number of irreducible representations and check it for it for the algebra of twisted polynomials, the quantum Weyl algebra and the algebra of regular functions on quantum group.

1 Introduction and main statements

Quantum algebras appear in the framework of mathematical physics. From the algebraic point of view a quantum algebra R_q is an domain and a free $\mathbb{C}[q, q^{-1}]$-module. After specialization module $q-\varepsilon$ we obtain the algebra $R_\varepsilon = R_q \bmod (q-\varepsilon)$. As usual (see algebras A1-A4 below) R_ε is a domain and, if ε is a root of unity, then R_ε is finite over its center Z_ε. We call R_ε a quantum order (since it becomes an order in the skew field $R_\varepsilon \otimes \mathrm{Fract}(Z_\varepsilon)$). This algebra defines the affine variety $\mathfrak{X} = \mathrm{Maxspec} Z_\varepsilon$ that is singular in general.

The order R_ε has one remarkable property: it admits the quantum adjoint action. For $a, u \in R_q$ we denote $a_\varepsilon, u_\varepsilon := a, u \bmod (q-\varepsilon) \in R_\varepsilon$. If u_ε lies in Z_ε, then the formula

$$\mathcal{D}_u(a_\varepsilon) = \frac{ua - au}{q-\varepsilon} \bmod (q-\varepsilon)$$

defines the derivation $\mathcal{D}_u : R_\varepsilon \to R_\varepsilon$, that is called the quantum adjoint action of u [1, 2, 4, 10, 11]. The center Z_ε is a Poisson algebra with respect to the bracket $\{u_\varepsilon, v_\varepsilon\} := \mathcal{D}_u(v_\varepsilon) = -\mathcal{D}_v(u_\varepsilon)$. The variety $\mathfrak{X} = \mathrm{Maxspec} Z_\varepsilon$ is a Poisson variety. It splits into symplectic leaves [7].

For a point $\chi \in \mathfrak{X} = \mathrm{Maxspec} Z_\varepsilon$, we consider the finite dimensional algebra $R_\chi = R_\varepsilon / R_\varepsilon m_\chi$. Call it the fiber of R_ε. An irreducible representation of R_ε with central character χ passes through $R_\varepsilon \to R_\chi$. Therefore, these representations are in one to one correspondence with irreducible representations of R_χ

The goal of this paper is to characterize the fibers in terms of a point χ of Poisson variety $\mathfrak{X}$. Main Theorem will be proved in the case when R_q is one of the following algebras: A1) Algebra of twisted polynomials, A2) Quantum Weyl Algebra, A3) $U_q(\mathfrak{b})$ (this algebra is isomorphic to $\mathbb{C}_q[B]$ for the Borel subroup B), A4) algebra $\mathbb{C}_q[G]$ of regular functions on the quantum semisimple Lie group G. For definitions see e.g. [6]

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We introduce the notion of stabilizer for any point of commutative associative Poisson $\mathbb{C}$-algebra $\mathcal{F}$. Recall that an algebra $\mathcal{F}$ is a Poisson algebra if it admits a Poisson bracket that is a linear, skew-symmetric map $\{\cdot,\cdot\} : \mathcal{F}\otimes\mathcal{F}\to\mathcal{F}$, subjecting to the Jacobi and the Leibniz identity (i.e., $\{a,bc\} = \{a,b\}c + b\{a,c\}$ for all $a,b,c\in\mathcal{F}$). A Poisson algebra is a Lie algebra with respect to the Poisson bracket. We call P an ideal(resp. Poisson ideal) of Poisson algebra $\mathcal{F}$ if P is an ideal of commutative associative algebra $\mathcal{F}$ (resp. P is an ideal of $\mathcal{F}$ and $\{P,\mathcal{F}\}\subset P$). We identify a point $\chi\in\mathfrak{X} = \mathrm{Maxspec}\mathcal{F}$ with the character $\chi:\mathcal{F}\to\mathbb{C}$. We use the notation m_χ for the corresponding maximal ideal in $\mathcal{F}$. The subalgebra

$$G_\chi := \{a\in\mathcal{F}:\ \{a,\mathcal{F}\}\in m_\chi\}$$

is a Poisson subalgebra of $\mathcal{F}$. The ideal m_χ^2 is contained in G_χ and is a Poisson ideal in G_χ.
Definition 1.1. The finite dimensional Lie $\mathbb{C}$-algebra

$$\mathfrak{g}_\chi := G_\chi/m_\chi^2$$

is called the stabilizer of the point $\chi\in\mathfrak{X}$. If $\mathcal{F}$ is generated (as commutative associative $\mathbb{C}$-algebra) by $a_1,\ldots,a_N$, then $\mathfrak{g}$ is the linear span of $\overline{a_i} := a_i - \chi(a_i) \bmod m_\chi^2$. The definition of stabilizer in the case of smooth manifolds is given in [8].

If $\mathfrak{g}$ is a finite dimensional Lie algebra over $\mathbb{C}$ and $\mathfrak{n}$ is the maximal nilpotent ideal (i.e., nilradical) of $\mathfrak{g}$, then the Lie algebra $\mathfrak{g}/\mathfrak{n}$ is a reductive algebra. We denote by $\mathrm{rank}\mathfrak{g}$ the dimension of maximal commutative subalgebra in $\mathfrak{g}/\mathfrak{n}$. If $\mathfrak{g}$ is an algebraic solvable Lie algebra (i.e., the Lie algebra of some algebraic solvable $\mathbb{C}$-group $\mathfrak{G}$), then $\mathfrak{g} = \mathfrak{t}\oplus\mathfrak{n}$ where $\mathfrak{t}$ is a toroidal Lie subalgebra of $\mathfrak{g}$. In this case $\dim_{\mathbb{C}}\mathfrak{t} = \mathrm{rank}\mathfrak{g}$.

Recall the definition of Ore extension. Given an algebra A; an automorphism $\tau: A\to A$ and a τ-derivation $\delta: A\to A$ (i.e. $\delta(ab) = \delta(a)b + \tau(a)\delta(b)$ for all $a,b\in A$). An algebra R is an Ore extension of A if R is generated by A and an indeterminate x with the defining relations $xa = \tau(a)x + \delta(a)$ for all $a\in A$ [6, 12].

Let we have a skew-symmetric integer matrix $\mathbb{S} = (s_{ij})_{ij=1}^N$ and an indeterminate q. Put $q_{ij} = q^{s_{ij}}$ and form the matrix $\mathbb{Q} = (q_{ij})_{ij}^n$. By definition, an algebra R_q is a quantum solvable algebra over $C := \mathbb{C}[q,q^{-1}]$ if it is generated by the elements $x_1,x_2,\ldots,x_n,x_{n+1}^{\pm1},\ldots,x_N^{\pm1}$ and C and such that any its subalgebra

$$R_i := < x_i,\ldots,x_n,x_{n+1}^{\pm1},\ldots,x_N^{\pm1},C >,\quad 1\le i\le n$$

is an Ore extension $R_i = R_{i+1}[x_i;\tau_i,\delta_i])$ with $\tau(x_j) = q_{ij}x_j,\quad i+1\le j\le N$ and $x_ix_j = q_{ij}x_jx_i$, $1\le i\le N,\ n+1\le j\le N$ [9, 11]. All algebras A1-A4 are quantum solvable algebras (see [9, 11]). More precisely, the algebra $\mathbb{C}_q[G]$ becomes quantum solvable after some localization (see section 3). In what follows we suppose that all quantum solvable algebras obey the following conditions.
1) "q-skew condition": $\tau_i\delta_i = q^{s_i}\delta_i\tau_i$ with some $s_i\in\mathbb{Z}$; assume that $s_i\ne0$ for $\delta_i\ne0$ and $s_i=0$ for $\delta_i=0$. Call the system of non-zero integers $\{s_i\}$ the system of exponents of R_q;
2) All δ_i is locally nilpotent.
Notice that the algebras A1-A4 obey these two conditions.

Let ε be a primitive lth root of unity. For the algebras A1-A2 we call l (and ε) admissible if if l is relatively prime with all principal minors of $\mathbb{S}$ and with the system of exponents $s_1,\ldots,s_N$. For the algebras A3-A4 l(and ε) is admissible if l is odd and $l\ge3$ in the case G has G_2 components.

If R_q is one of the algebras A1-A4 and l is an admissible root of unity, then the elements $a_i := x_{i,\varepsilon}^l$ $1\le i\le N$ (here $x_{i,\varepsilon} = x_i \bmod (q-\varepsilon)$) lie in the center Z_ε of R_ε [11, Lemma 2.19].

The central subalgebra Z_0, generated by $a_i,\ 1 \le i \le n$ and $a_i^{-1},\ n+1 \le i \le N$, is isomorphic to $\mathbb{C}[a_1, \ldots, a_n, a_{n+1}^{\pm 1}, \ldots a_N^{\pm 1}]$. The subalgebra Z_0 is called the l-center of R_ε. For all algebras A1-A4 the subalgebra Z_0 is a Poisson subalgebra of Z_ε. The embedding $Z_0 \subset Z_\varepsilon$ defines the projection $\phi : \mathfrak{X} \to \mathfrak{X}_0$ where $\mathfrak{X}_0 = \mathrm{Maxspec} Z_0$.

The goal of this paper is to prove the following statement for the algebras A1-A4.

Main Theorem. Let R be one of the algebras A1-A4. Suppose that l is an admissible. Let $\phi : \chi \mapsto \chi_0$ and $\mathfrak{g}_\chi$ (resp. $\mathfrak{g}_{\chi_0}$) are stabilizer of χ(resp. χ_0). Then

1) $\mathfrak{g}_\chi$ and $\mathfrak{g}_{\chi_0}$ are algebraic solvable Lie algebras with decompositions $\mathfrak{g}_\chi = \mathfrak{t} \oplus \mathfrak{n}$ and $\mathfrak{g}_{\chi_0} = \mathfrak{t}_0 \oplus \mathfrak{n}_0$.

2) the subalgebra G_{χ_0} (of the algebra Z_0) is contained in G_χ; the embedding $G_{\chi_0} \subset G_\chi$ is extended to the homomorphism $\psi_p : \mathfrak{g}_{\chi_0} \to \mathfrak{g}_\chi$ such that its restriction on $\mathfrak{t}_0$ is an isomorphism $\psi_p|_{\mathfrak{t}_0} : \mathfrak{t}_{\chi_0} \to \mathfrak{t}_\chi$; $\mathrm{rank}\mathfrak{g}_\chi = \mathrm{rank}\mathfrak{g}_{\chi_0}$;

3) the number $|\mathrm{IrrR}_\chi|$ of irreducible representations of R_ε with central character χ is equal to $l^{\mathrm{rank}\mathfrak{g}_\chi}$.

In the section 2 we prove Main Theorem in the partial case (Proposition 2.2) and show that Main Theorem is true for the quantum solvable algebras with "admissible stratification" (Proposition 2.5). In the next section 3 we prove the existence of admissible stratifications for the algebras A1-A4. This will conclude the proof of Main Theorem for the algebras A1-A4 (see Propositions 3.1-3.4).

2 Standard ideals

Let $\mathbb{S} = (s_{ij})_{ij=1}^N$, $C := \mathbb{C}[q, q^{-1}]$ be as above. We denote by $A_\mathbb{Q}$ the algebra of twisted polynomials (see section 3).

Consider a quantum solvable algebra $\mathfrak{R}'$ over C, generated by

$$x_1, \ldots, x_m, x_{m+1}, \ldots, x_N.$$

Suppose that the elements $x_1, \ldots, x_m$ are q-commute, i.e. $x_i x_j = q_{ij} x_j x_i,\ 1 \le i, j \le m$. The multiplicatively closed subset $\mathfrak{S}$, generated by $x_1, \ldots, x_m$, is a denominator subset [17, Lemma 2.1]. Denote $\mathfrak{R} := \mathfrak{R}'\mathfrak{S}^{-1}$. There exist the elements $\tilde{x}_{m+1}, \ldots, \tilde{x}_N$ in the localization of $\mathfrak{R}S_*^{-1}$ (here S_* is finitely generated, by some $\{q^i - 1\}$, denominator subset in C) such that $x_i \tilde{x}_j = q_{ij} \tilde{x}_j x_i,\ 1 \le i \le m,\ m+1 \le j \le N$. The subalgebra, generated by $\tilde{x}_{m+1}, \ldots, \tilde{x}_N$ coincides with $\mathfrak{R}_{m+1} =< x_{m+1}, \ldots, x_N >$ [17, Prop.2.1-2.3]. Suppose that the ideal $\mathfrak{I}'$ (of $\mathfrak{R}S_*^{-1}$), generated by $\tilde{x}_j,\ m+1 \le j \le N$, has zero intersection with C. Denote $\mathfrak{I} := \mathfrak{I}' \cap \mathfrak{R}$. Call $(\mathfrak{R}, \mathfrak{I})$ a standard pair and $\mathfrak{I}$ a standard ideal in $\mathfrak{R}$. Notice that the subalgebra $\mathfrak{B}$, generated over C by $x_i, 1 \le i \le m$, is an algebra of twisted Laurent polynomials and $\mathfrak{B} = \mathfrak{R}/\mathfrak{I}$.

Let ε be a primitive lth root of unity and l is relatively prime with all principal minors of $\mathbb{S}$. We denote by $\mathfrak{Z}_\varepsilon$ the center of $\mathfrak{R}_\varepsilon := \mathfrak{R} \bmod (q - \varepsilon)$ and $\mathfrak{X} := \mathrm{Maxspec}\mathfrak{Z}_\varepsilon$. The elements $a_i := x_{i,\varepsilon}^l, 1 \le i \le m$ lie in $\mathfrak{Z}_\varepsilon$. Let $\mathfrak{Z}_0$ be some subalgebra of $\mathfrak{Z}_\varepsilon$ such that $\mathfrak{Z}_0 \cap \mathfrak{B}_\varepsilon$ is generated by $a_i := x_{i,\varepsilon}^l, 1 \le i \le m$ and $\mathfrak{R}_\varepsilon$ is finite over $\mathfrak{Z}_0$. The center $\mathfrak{Z}_\varepsilon$ is finite over $\mathfrak{Z}_0$. Denote $\mathfrak{X}_0 := \mathrm{Maxspec}\mathfrak{Z}_0$, $\mathfrak{I}_\varepsilon = (\mathfrak{I} + \mathfrak{R}(q - \varepsilon)) \bmod (q - \varepsilon)$, $\phi : \mathfrak{X} \to \mathfrak{X}_0$, $\mathfrak{i} := \mathfrak{I}_\varepsilon \cap \mathfrak{Z}_\varepsilon$, $\mathfrak{i}_0 := \mathfrak{I}_\varepsilon \cap \mathfrak{Z}_0$.

The skew field $\mathrm{Fract}(\mathfrak{R})$ is isomorphic to the skew field $\mathrm{Fract}(A_\mathbb{Q})$ of the algebra $A_\mathbb{Q}(=\mathrm{gr}(\mathfrak{R}))$ of twisted polynomials (see section 3) [9, 11, 17]. We are going to prove Main Theorem for the case χ (resp.χ_0) is a point of $\mathfrak{X}$ (resp. $\mathfrak{X}_0$) annihilated by $\mathfrak{i}$ (resp.$\mathfrak{i}_0$).

The algebra $\mathfrak{B}$ has a new system of generators $h_i, g_i,\ 1 \le i \le k$ and $z_j,\ 1 \le j \le p$, $2k + p = m$ (that consists of monomials of $x_i^{\pm 1}$, $1 \le i \le m$) such that $h_i g_i = q^{d_i'} g_i h_i$ and $\{z_j\}$ generate the center of $\mathfrak{B}$. By assumption, l is relatively prime with d_i'. The intersection of the center $\mathfrak{Z}$ of $\mathfrak{R}$ with $\mathfrak{B}$ is generated by some monomials $\{z^a := z_1^{\alpha_1} \cdots z_p^{\alpha_p},\ \alpha_j \in \mathbb{Z}\}$. Choosing the

compatible basis, we may consider that the intersection $\mathfrak{Z}\cap\mathfrak{B}$ is generated by $z_{t+1}^{n_{t+1}},\dots,z_p^{n_p}$ for some $n_{t+1},\dots,n_p\in\mathbb{N}$. Since the field Center(Fract($(\mathrm{A}_{\mathbb{Q}})$)) is algebraically closed in Fract$(A_{\mathbb{Q}})$, then if an element z^d also lies in the center of Fract$(A_{\mathbb{Q}})$(that is isomorphic to Fract$(\mathfrak{R})$), then z lies in the center. This verifies that $n_{t+1}=\dots=n_p=1$.

Lemma 2.1.

1) The intersection $\mathfrak{Z}_\varepsilon\cap\mathfrak{B}_\varepsilon$ is generated by

$$h_{i,\varepsilon}^l,\quad g_{i,\varepsilon}^l,\ \text{where } 1\le i\le k \text{ and } z_{1,\varepsilon}^l,\dots,z_{t,\varepsilon}^l,\ z_{t+1,\varepsilon},\dots,z_{m,\varepsilon}.$$

2) The intersection $\mathfrak{Z}_0\cap\mathfrak{B}_\varepsilon$ is generated by

$$h_{i,\varepsilon}^l,\quad g_{i,\varepsilon}^l \text{ where } 1\le i\le k \text{ and } z_{1,\varepsilon}^l,\dots,z_{t,\varepsilon}^l,\ z_{t+1,\varepsilon}^l,\dots,z_{m,\varepsilon}^l.$$

Proof. The statement 2) is trivial. To prove 1) it suffices to show that the monomial $z_{1,\varepsilon}^{\alpha_1}\dots z_{t,\varepsilon}^{\alpha_t}$ lies in $\mathfrak{Z}_\varepsilon$ whenever l divides all α_i.

There exists the system of generators $\tilde{x}_{k_1},\dots,\tilde{x}_{k_t}$ in $A_{\mathbb{Q}}$ such that

$$z_j\tilde{x}_{k_j}=q^{\nu_{i,k_j}}\tilde{x}_{k_j}z_j\quad \text{and } \mathrm{F}:=\det(\nu_{\mathrm{i,k_j}})_{\mathrm{i,j=1}}^{\mathrm{t}}\neq 0.$$

Connect $\tilde{x}_{k_1},\dots,\tilde{x}_{k_t}$ to the system $\{x_i, 1\le i\le m\}$. Denote by $\mathbb{S}''$ the corresponding $(m+t)\times(m+t)$-submatrix of $\mathbb{S}$. The rank of $\mathbb{S}''$ is equal to $2k+2t$ and the greatest common divisor D'' of all its $(2k+2t)\times(2k+2t)$-minors is equal to $(d_1')^2\dots(d_k')^2F^2$. Since l is admissible, l relatively prime with D''. Therefore, $GCD(l,F)=1$. There exist $v_i\in$ Fract$(\mathfrak{R})$ such that

$$z_iv_j=q^{p_i\delta_{ij}}v_jz_i,\ \text{and } \mathrm{GCD}(l,p_i)=1$$

for all $1\le i,j\le t$. This implies that, if $z_{1,\varepsilon}^{\alpha_1}\dots z_{t,\varepsilon}^{\alpha_t}$ lies in $\mathfrak{Z}_\varepsilon$, then l divides all α_i.□

Proposition 2.2. Let $\mathfrak{R}$, $\mathfrak{B}$, $\mathfrak{I}$, ε be as above. Suppose that $\mathfrak{Z}_0$ is a Poisson subalgebra in $\mathfrak{Z}_\varepsilon$. Let $\chi\in\mathfrak{X}$ and $\chi_0=\phi(\chi)\in\mathfrak{X}_0$. Suppose that χ (resp. χ_0) is annihilated by the ideal $\mathfrak{i}$ (resp.$\mathfrak{i}_0$) and $\chi(a_i)\neq 0$, $1\le i\le m$. Then

1) the number of irreducible representations of $\mathfrak{R}_\varepsilon$ with central character χ is equal to l^t;

2) the ideal $\mathfrak{i}$ (resp. $\mathfrak{i}_0$) is a Poisson ideal in G_χ (resp. G_{χ_0}). Denote by $\mathfrak{n}'$ (resp. $\mathfrak{n}_0'$) the image of $\mathfrak{i}$ (resp. $\mathfrak{i}_0$) in $\mathfrak{g}_\chi$ (resp. $\mathfrak{g}_{\chi_0}$);

3) the ideal $\mathfrak{n}'$ (resp. $\mathfrak{n}_0'$) is a nilpotent ideal in $\mathfrak{g}_\chi$ (resp. $\mathfrak{g}_{\chi_0}$). Then Main Theorem is true for $\mathfrak{R}$ and χ. In particular, $\mathfrak{g}_\chi$ (resp. $\mathfrak{g}_{\chi_0}$) is an algebraic solvable Lie algebra.

Proof. First, notice that $\mathfrak{I}$ lies in the radical of $\mathfrak{R}_\varepsilon\mathfrak{i}_0$ (apply [11, Lemma 5.1]). Kernel of any irreducible representation π with l-central character χ_0 contains $\mathfrak{I}$. Any irreducible representation with l-central character $\chi_0\in\mathfrak{X}_0$ is uniquely determined by its kernel, generated by

$$\mathfrak{i},\quad h_{i,\varepsilon}^l-\chi(h_{i,\varepsilon}^l),\ g_{i,\varepsilon}^l-\chi(g_{i,\varepsilon}^l) \text{ for } 1\le i\le k,$$

$$z_{j,\varepsilon}-\chi(z_{j,\varepsilon}) \text{ for } 1\le j\le t,\quad z_{j,\varepsilon}-\chi(z_{j,\varepsilon}) \text{ for } t+1\le j\le p.$$

The number of irreducible representations with central character χ is equal to l^t. This proves 1).

To calculate subalgebras $\mathfrak{g}_\chi$ and $\mathfrak{g}_{\chi_0}$ we find generators of the subalgebras G_χ and G_{χ_0} of $\mathfrak{R}_\varepsilon$:

$$G_\chi=<z_{j,\varepsilon}^l-\chi(z_{j,\varepsilon}^l), 1\le j\le t;\ z_{j,\varepsilon}-\chi(z_{j,\varepsilon}), t+1\le j\le p;\ \ \mathfrak{i}>,$$

$$G_{\chi_0}=<z_{j,\varepsilon}^l-\chi(z_{j,\varepsilon}^l), 1\le j\le p;\ \mathfrak{i}_0>.$$

We see $G_{\chi_0} \subset G_\chi$. This defines the homomorphism $\psi : \mathfrak{g}_{\chi_0} \to \mathfrak{g}_\chi$.

Since $\mathfrak{i}$ is an intersection of $\mathfrak{R}$-ideal $\mathfrak{I}$ with $\mathfrak{Z}_\varepsilon$, then $\mathfrak{i}$ is a Poisson ideal of $\mathfrak{Z}_\varepsilon$ [10, Lemma 3.12]. Similar for $\mathfrak{i}_0$. This implies 2). Denote

$$e_i = z^l_{i\varepsilon} - \chi(z^l_{i\varepsilon}) \bmod m^2_\chi, \quad 1 \le i \le t \text{ and } \mathfrak{t} = \mathrm{span}\{\mathrm{e_i};\ 1 \le \mathrm{i} \le \mathrm{t}\}.$$

The Lie algebra $\mathfrak{g}_\chi$ is a sum (as a linear space) of $\mathfrak{t}$ and $\mathfrak{n}$ spanned modulo m^2_χ by $z_{j,\varepsilon} - \chi(z_{j,\varepsilon})$, $t+1 \le j \le p$ and $\mathfrak{n}'$. Similarly, $\mathfrak{g}_{\chi_0}$ is a sum (as a linear space) of $\mathfrak{t}_0 = \mathfrak{t}$ and $\mathfrak{n}_0$, spanned modulo $m^2_{\chi_0}$ by $z^l_{j,\varepsilon} - \chi(z^l_{j,\varepsilon})$, $t+1 \le j \le p$ and $\mathfrak{n}'_0$.

Let us prove that $\mathfrak{n}$ is a nilpotent ideal in $\mathfrak{g}_\chi$. Similar for $\mathfrak{n}_0$. Any element of $\mathfrak{R}S_*^{-1}$ (is a sum of monomials of

$$x_1^{n_1} \cdots x_m^{n_m} \tilde{x}_{m+1}^{n_{m+1}} \cdots \tilde{x}_N^{n_N}$$

We define the degree $\deg(a) := (n_{m+1}, \ldots, n_N)$ of any monomial a. For any two monomials a, b there exists $s \in \mathbb{Z}$ such that $ab - q^s ba$ is a sum of monomials of lower degree with respect to the lexicographical ordering. For any $A, B \in Z_\varepsilon$ we have $\{A, B\} = \mathrm{const}AB + \{\textit{the lower terms}\}$. This verifies that $\mathfrak{n}$ is a nilpotent ideal.

Let us prove that the Lie subalgebra $\mathfrak{t}$ is diagonalizable. The elements $x_1, \ldots, x_m$ are FA-elements in $\mathfrak{R}$ [9, 11]. That is for any $1 \le i \le m$ and $a \in \mathfrak{R}$ there exists a polynomial $f(t)$ (with roots in $\{q^s\}_{s\in\mathbb{Z}}$) such that $f(\mathrm{Ad}_{x_i}(a) = 0$. The adjoint action Ad_{x_i} is diagonalizable [9]. One can choose $f(t)$ with different roots $q^{\gamma_1}, \ldots, q^{\gamma_k}$. The derivation $\mathcal{D}'_i := x_\varepsilon^{-l}\mathcal{D}_{x^l} : \mathfrak{R}_\varepsilon \to \mathfrak{R}_\varepsilon$ for $x = x_i$ obey $f_1(\mathcal{D}'_i)(a_\varepsilon) = 0$ where $f_1(t)$ is a polynomial with different roots $c\gamma_1, \ldots, c\gamma_k$, $c = l\varepsilon^{l-1}$. This imply that $\mathcal{D}'_i$ is diagonalizable. The same is true for z^l_i. Finally, ad_{e_i} are simultaneously diagonalizable.□

Definition 2.3. Let R be domain with unit. Consider the set of pairs $\{(\mathcal{P}_\mu, S_\mu)\}$ where S_μ is a denominator subset R and $\mathcal{P}_\mu$ is a prime ideal in R (i.e $\mathcal{P}_\mu \in \mathrm{Spec}(R)$) with empty intersection with S_μ. We call $\{(\mathcal{P}_\mu, S_\mu)\}$ a stratification of $\mathrm{Spec}(R)$ if for any $I \in \mathrm{Spec}(R)$ there exists a unique μ such that $I \supset \mathcal{P}_\mu$ and $I \cap S_\mu = \emptyset$. If R is a free C-module over commutative ring C, we assume, in addition, that I and any $\mathcal{P}_\mu$ have zero intersection with C.

Definition 2.4. Let R_q be a quantum solvable algebras over $C := \mathbb{C}[q, q^{-1}]$ and $\{(\mathcal{P}_\mu, S_\mu)\}$ be a stratification of R_q. We call $\{(\mathcal{P}_\mu, S_\mu)\}$ an admissible stratification if

1) for any μ there exists isomorphism $\theta_\mu : R_q S_\mu^{-1} \to \mathfrak{R}_\mu$ such that $\mathfrak{R}_\mu$ and $\mathfrak{I}_\mu := \theta_\mu(\mathcal{P}_\mu)$ form a standard pair;

2) the stratification $\{(\mathcal{P}_\mu, S_\mu)\}$ admits specialization modulo $q - \varepsilon$ (i.e. $\{(\mathcal{P}_{\mu,\varepsilon}, S_{\mu,\varepsilon})\}$ is a stratification of $\mathfrak{R}_\varepsilon$).

3) $S_{\mu,\varepsilon} := S_\mu \bmod (q - \varepsilon) \subset Z_0$ and $\theta_\mu(S_\mu)$ is generated by $x^l_1, \ldots, x^l_m$.

Proposition 2.5. Let R_q be a quantum solvable algebra and l be admissible for R_q. Suppose that $x^l_{1,\varepsilon}, \ldots, x^l_{N,\varepsilon}$ lie in the center of R_ε and generate a Poisson central subalgebra (denote Z_0). Suppose that R_q has an admissible stratification $\{(\mathcal{P}_\mu, S_\mu)\}$. Then Main Theorem is true for χ.

Proof. Let $\chi \in \mathfrak{X}$. Choose μ such that $\chi(S_{\mu,\varepsilon}) \neq 0$ and χ is annihilated by $\mathfrak{i}_\mu$. Apply Proposition 2.2. □

3 Existence of admissible stratification

To prove Main Theorem we present an admissible stratification for quantum algebras A1-A4.

A1) The algebra of twisted polynomials. Let the matrices $\mathbb{Q}$ and $\mathbb{S}$ be as in the above. The algebra $R = A_\mathbb{S}$ of twisted polynomials is generated by $x_1, \ldots x_n, x^{\pm 1}_{n+1}, \ldots, x^{\pm 1}_N$ subject to the relations $x_i x_j = q_{ij} x_j x_i$.

Choose some subset $T \subset \Lambda = \{1, 2, \ldots, n\}$. Consider the ideal $\mathcal{P}_T$ generated by $\{x_i : i \in T\}$ and the denominator subset S_T generated by $\{x_i^l : i \notin T\}$.

Proposition 3.1. Main Theorem is true for the Algebra of twisted polynomials.

Proof. The set of pairs $\{(\mathcal{P}_T, S_T)\}$ is an admissible stratification. By direct calculations, $\{a_i, a_j\} = cs_{ij}a_ia_j$ where $a_i = x_{i,\varepsilon}^l$ and $c = l\varepsilon^{l-1}$. Apply Proposition 2.5. □

A2) Quantum Weyl algebra. Let $\mathbb{S} = (s_{ij})_{ij}^n$ be skew-symmetric integer matrix and q be indeterminate. As above we put $q_{ij} = q^{s_{ij}}$ and form the matrix $\mathbb{Q} = (q_{ij})_{ij=1}^n)$. Given non-zero integers $s_1, \ldots, s_n$ define

$$q_1 = q^{s_1}, \ldots, q_n = q^{s_n}.$$

We consider two new matrices. The first matrix $\mathbb{P} = ((p_{ij})_{ij=1}^n)$ with entries subject $p_{ii} = p_{ij}p_{ji} = 1$ and such that

$$p_{ij} = q_i q_{ij}, \text{ for i < j.}$$

The second one $\mathbb{R} = (r_{ij})_{ij=1}^n)$ has entries

$$r_{ij} = \begin{cases} q_{ji}, \text{ if i < j,} \\ q_i, \text{ if i = j,} \\ p_{ji} = q_j q_{ji}, \text{ if i > j.} \end{cases}$$

Form skew-symmetric integer matrix $\mathbb{T} = (t_{ij})_{ij=1}^n$ such that $p_{ij} = q^{t_{ij}}$ and integer matrix $\mathbb{U} = (u_{ij})_{ij=1}^n$ such that $r_{ij} = q^{u_{ij}}$. Form matrices

$$\mathbb{Q}^* = \begin{pmatrix} \mathbb{Q} & -\mathbb{R} \\ \mathbb{R} & \mathbb{P} \end{pmatrix}, \quad \mathbb{S}^* = \begin{pmatrix} \mathbb{S} & -\mathbb{U} \\ \mathbb{U} & \mathbb{T} \end{pmatrix}.$$

Definition 3.2. The Quantum Weyl algebra W is generated by $y_1, \ldots, y_n, x_1, \ldots, x_n$ with the following relations $y_iy_j = q_{ij}y_jy_i$, $x_ix_j = p_{ij}x_jx_i$, $x_iy_j = r_{ij}y_jy_i$ for $i \neq j$ and

$$x_iy_i = q_iy_ix_i + \sum_{k<i}(q_k - 1)y_kx_k + 1. \tag{3.1}$$

The algebra W is an quantum solvable algebra over $\mathbb{C}[q, q^{-1}]$ with the system of exponents $s_1, \ldots, s_n$. Denote $h_k = y_kx_k$. The relations imply

$$\begin{pmatrix} x_i \\ y_i \end{pmatrix} h_k = h_k \begin{pmatrix} q_ix_i \\ q_i^{-1}y_i \end{pmatrix} \quad \text{for } i < k,$$

$$\begin{pmatrix} x_i \\ y_i \end{pmatrix} h_k = h_k \begin{pmatrix} x_i \\ y_i \end{pmatrix} \quad \text{for } i > k.$$

For any $1 \leq i \leq n$ we denote $w_i = 1 + \sum_{k\leq i}(q_k - 1)y_kx_k$.

One can rewrite (3.1) as follows $x_iy_i = q_iy_ix_i + w_{i-1}$. The variables x_i, y_i, w_i obey the relations

$$y_iw_j = \begin{cases} q_i^{-1}w_jy_i, \text{ for } i \leq j, \\ w_jy_i, \text{ for } i > j, \end{cases} \quad , \quad x_iw_j = \begin{cases} q_iw_jx_i, \text{ for } i \leq j, \\ w_jx_i, \text{ for } i > j, \end{cases} .$$

By definition, ε is an admissible lth root of unity if l is relatively prime with all principal minors of S^* and with $s_1, \ldots, s_n$. The elements $a_i := x_{i,\varepsilon}^l, b_i := y_{i,\varepsilon}^l$ lie in the center of Z_ε of W_ε and generate the central subalgebra Z_0.

Denote $f_i = w_{i,\varepsilon}^l$, $1 \leq i \leq n$. Similarly to [13], one can prove that there exists the chain of non-zero complex numbers $\gamma_1, \ldots, \gamma_{n-1}$ such that

$$f_i = 1 + \sum_{k<i} \gamma_k a_k b_k.$$

This implies that $f_i \in Z_0$. By direct calculations, $\{a_i, a_j\} = \gamma t_{ij} a_i a_j$, $\{b_i, b_j\} = \gamma s_{ij} b_i b_j$, $\{a_i, b_j\} = \gamma u_{ij} a_i b_j$, $\{a_i, b_i\} = \gamma s_i a_i b_i + f_{i-1}$ where $\gamma = l^2 \varepsilon^{-1}$. We see that Z_0 is a Poisson subalgebra in Z_ε.

Denote $\Lambda = \{1, \ldots, n\}$. We shall call a triple $T = (T_1, T_2, T_3)$ of subsets of Λ an admissible triple if $T_1 \subseteq T_2 \subseteq T_3$ and the following property holds: if $i \in T_2$ then i and $i-1 \in T_3$.

Consider the ideal $\mathcal{P}_T$ of W generated by x_i, y_j and w_k with $i \in T_1$, $j \in T_2$ and $k \in T_3$. Form the denominator subset S_T generated by the following q-commuting elements $\{x_i^l, i \in T_2 - T_1\}$, $\{y_i^l, i \in \Lambda - T_2\}$ and $\{w_i^l, i \in \Lambda - T_3\}$. The subset S_T has empty intersection with $\mathcal{P}_T$. The set of pairs $\{(\mathcal{P}_T, S_T)\}$ is a stratification of W (see [15, 16]).

Proposition 3.3. Main Theorem is true for the Quantum Weyl algebra.

Proof. The set of pairs $(\mathcal{P}_T, S_T)$ is an admissible stratification. □

A3-A4. Cases of algebras $U_q(\mathfrak{b}) = \mathbb{C}_q[B]$ and $\mathbb{C}_q[\mathrm{G}]$. Let $\mathfrak{g}$ be a semisimple Lie algebra with the system of simple roots $\alpha_1, \ldots, \alpha_n$. Let G be its simply connected Lie group. Denote $d_i := \frac{(\alpha_i, \alpha_i)}{2}$ and $C := \mathbb{C}[q, q^{-1}, (q^{d_i} - q^{-d_i})^{-1}]$. The quantum universal enveloping algebra is an Hopf algebra over C generated by $E_i, F_i, K_i^{\pm 1}$, $1 \leq i \leq n$ obeying Drinfeld-Jimbo relations. The algebra $\mathbb{C}_q[\mathrm{G}]$, the subalgebra of the dual Hopf algebra for $U_q(\mathfrak{g})$, is generated by matrix entries of irreducible finite dimensional representations $c_{f,v}(a) := f(av)$, $v \in V$, $f \in V^*$, $a \in U_q(\mathfrak{g})$.

We assume that l is admissible. In the case of algebras A3-A4: l is admissible if l is odd and $l \geq 3$ in the case G has G_2 components. The algebra $C_\varepsilon[\mathrm{G}]$ has a central Poisson subalgebra Z_0 that is isomorphic to $\mathbb{C}[G]$ with the standard Belavin-Drinfeld bracket [3]. The algebra $\mathbb{C}_q[\mathrm{G}]$ has a subalgebra R_q^+ generated by matrix entries $c_{f,v_{-\lambda}}$ where $v_{-\lambda}$ is the vector of lowest weight in the irreducible representation V_λ with highest weight λ. The algebra R_q^+ is isomorphic to $\mathbb{C}_q[\mathrm{B}]$ where $\mathrm{B} := \mathrm{B}^+$. By Drinfeld pairing the algebra $\mathbb{C}_q[\mathrm{B}]$ is isomorphic to $U_q(\mathfrak{b}^-)$.

Proposition 3.4. Main theorem is true for the algebras A3-A4.

Proof. First, notice that the algebra $\mathbb{C}_q[\mathrm{G}]$ has the denominator subset S generated by matrix entry $c_{\rho, v_{-\rho}}$ where ρ equal to the half of sum of positive roots. The localization $\mathbb{C}_q[\mathrm{G}]S^{-1}$ is isomorphic to the subalgebra in $U_q(\mathfrak{b}^-) \otimes U_q(\mathfrak{b}^+)$ generated by $K_\lambda \otimes K_{-\lambda}$, $F_i \otimes 1$, $1 \otimes E_i$, $1 \leq i \leq n$. It suffices to construct an admissible stratification for $\mathbb{C}_q[\mathrm{B}]$.

The algebra R_q^+ (that is equal to $\mathbb{C}_q[\mathrm{B}] = U_q(\mathfrak{b}^-)$) has an stratification $(\mathcal{P}_w, S_w)$ where w is an element of the Weyl group W [3, 5, 18]. By definition, the ideal $\mathcal{P}_w$ is generated (as ideal) by the elements $c_{f,v_{-\lambda}}$ where f is orthogonal to subspace $U_q(\mathfrak{b}^-) t_w v_{-\lambda}$(here t_w is the corresponding element of the braid group). The denominator subset S_w is generated by the element $z_w := c_{wf^\rho, v_{-\rho}}$ where f^ρ is the element of highest weight in V_ρ^*.

Below we present the other construction of pair $(\mathcal{P}_w, S_w)$. Decompose the element w_0 (of highest length in the W) into product of simple reflections

$$w_0 = s_1 \ldots s_k s_{k+1} \ldots s_N, \quad s_t := s_{\alpha_{i_t}}$$

such that $w = s_1 \ldots s_k$. Denote $w_t = s_1 \ldots s_t$ (here $w_k = w$) and $z_t = z_{w_t}$. The elements z_i are q-commute [5, Cor. 3.2]. As usual denote $\beta_t := s_1 \ldots s_{t-1}(\alpha_{i_t})$. The algebra R_q^+ is a quantum solvable algebra with respect to the chain of generators

$$K_1^{\pm 1}, \ldots, K_n^{\pm 1}, F_{\beta_1}, \ldots, F_{\beta_1}, \ldots, F_{\beta_N}.$$

We denote by B_t the subalgebra generated by $K_1^{\pm 1}, \ldots, K_n^{\pm 1}, F_{\beta_k}, \ldots, F_{\beta_t}$. We obtain the filtration $B_1 \subset \ldots \subset B_k \subset B_N = R_q^+$. The subalgebra B_k depends only on w (denote $B_w := B_k$) [5]. The element z_t lies in B_t and don't lie in B_{t-1} [5, Lemma 3.2]. Denote by S_t the denominator subset generated by z_t. The ideal $\mathcal{P}_w$ has zero intersection with B_k and $B_k S_k^{-1} = R_q^+ S_k^{-1} / \mathcal{P}_w S_k^{-1}$. Let S_w be the denominator subset generated by S_t, $1 \leq t \leq k$ and

$\mathfrak{R}_w := R_q^+ S_w^{-1}$. The elements z_t are FA-elements in quantum solvable algebra $\mathfrak{R}_w$ [9, 11]. The adjoint action Ad_{z_i} are diagonalizable. Choose the new generators $z_1^{\pm 1}, \ldots, z_k^{\pm 1}, \tilde{F}_{\beta_{k+1}}, \ldots, \tilde{F}_{\beta_N}$ in localization $R_q^+ S_*^{-1}$ (for definition of S_* see section 2). The ideal $\mathcal{P}_w S_*^{-1}$ is generated by $\tilde{F}_{\beta_t}$ $k+1 \leq t \leq N$. The elements $z_{t,\varepsilon}^l$ lie in Z_0 [5, Theorem 1.6]. The pair $(R_q^+, \mathcal{P}_w)$ is a standard pair and the stratification $(\mathcal{P}_w, S_w)$ is an admissible stratification. This verifies the statement for $\mathbb{C}_q$[B] (and therefore for $\mathbb{C}_q$[G]).$\square$

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